

The number of colors in the decays $\pi^0, \eta, \eta' \rightarrow \gamma\gamma$

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Abstract

The decays $\pi^0, \eta, \eta' \rightarrow \gamma\gamma$ are investigated up to next-to-next-to leading order in the framework of the combined $1/N_c$ and chiral expansions. Without mixing of the pseudoscalar mesons the N_c independence of the π^0 and η decay amplitudes is shown to persist at the one-loop level, although the contribution of the Wess-Zumino-Witten term to the pertinent vertices is not canceled by the N_c dependent part of a Goldstone-Wilczek term. The decay amplitude of the singlet field, on the other hand, depends strongly on N_c and yields under the inclusion of mixing also a strong N_c dependence for the η decay. Both the η and η' decay are suited to confirm the number of colors to be $N_c = 3$.

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1 Introduction

Experimental evidence suggests that we live in a world with three colors. At high energies the Drell ratio for e^+e^- annihilation supports $N_c = 3$, while in the low-energy regime below 1 GeV the anomalous decay of the π^0 into two photons, $\pi^0 \rightarrow \gamma\gamma$, is presented as a textbook example to confirm the number of colors, see *e.g.* [1]. The quark charges are assumed to be independent of N_c yielding at tree level a width $\Gamma_{\pi^0 \rightarrow \gamma\gamma}$ proportional to N_c^2 , thus being quite sensitive to the number of colors. However, it was shown recently in [2] that the cancellation of triangle anomalies in the standard model with an arbitrary number of colors leads to N_c dependent values of the quark charges. For three light flavors (u, d, s) N_c enters as a quantized prefactor of the Wess-Zumino-Witten (WZW) term [3, 4], but the vertex with one pion and two photons is completely canceled by the N_c dependent part of a Goldstone-Wilczek term [2, 5]. A similar cancellation also occurs for the decay $\eta \rightarrow \gamma\gamma$, if one neglects η - η' mixing.

On the other hand, the quark triangle diagram of the microscopic theory describing the decay $\eta_0 \rightarrow \gamma\gamma$ of the flavor singlet is N_c dependent and hence due to η - η' mixing the decay width of the η will also pick up an N_c dependent portion. If one works with different up- and down-quark masses, $m_u \neq m_d$, the π^0 will also undergo mixing with the η - η' system and its decay width into two photons will have an—albeit small— N_c dependent piece.

Loop diagrams which have not been discussed in [2] can be another source of N_c -dependence for the two-photon decays. As we will see, the vertices of the one-loop diagrams contain indeed an N_c dependent piece that does not cancel out in the sum of the WZW and Goldstone-Wilczek terms.

In order to investigate systematically the effects of mixing and loops for the two-photon decays of π^0, η and η' , we include the η' explicitly within the combined framework of chiral perturbation theory (ChPT) and the $1/N_c$ expansion, so-called large N_c ChPT [6, 7, 8].² In this theory, the η_0 is combined with the octet of pseudoscalar mesons (π, K, η_8), since in the large N_c limit the axial $U(1)$ anomaly vanishes and the η' converts into a Goldstone boson. In the present work, we evaluate the decay amplitudes of π^0, η and η' up to next-to-next-to-leading order at which loops start contributing in large N_c ChPT.

The paper is organized as follows. In the next section, we discuss the WZW term under the inclusion of the η' . We will see that it can be decomposed into the conventional $SU(3)$ WZW Lagrangian, the Goldstone-Wilczek term and counter terms of unnatural parity. In Sec. 3 the calculation for the decays up to next-to-next-to-leading order is presented. Numerical results and the importance of N_c dependent contributions are discussed in Sec. 4. Sec. 5 contains our conclusions and the scaling behavior of the coupling constants under changes of the QCD running scale is presented in App. A.

2 Wess-Zumino-Witten term

In this section we will first briefly outline the method of extending the $SU(3)_R \times SU(3)_L$ chiral rotations of the effective Lagrangian in conventional ChPT to $U(3)_R \times U(3)_L$ in a more generalized framework including the η' [6, 8]. Within this approach the topological charge

²For an alternative approach to include the η' without employing large N_c counting rules, see, *e.g.*, [9].

operator coupled to an external field θ is added to the QCD Lagrangian

$$\mathcal{L} = \mathcal{L}_{QCD} - \frac{g^2}{16\pi^2} \theta(x) \text{tr}_c(G_{\mu\nu} \tilde{G}^{\mu\nu}) \quad (1)$$

with $G_{\mu\nu}$ the gluonic field strength tensor, $\tilde{G}_{\mu\nu} = \epsilon_{\mu\nu\alpha\beta} G^{\alpha\beta}$ its dual counterpart, and tr_c is the trace over the color indices. Under $U(1)_R \times U(1)_L$ the axial $U(1)$ anomaly adds a term $-\frac{g^2}{16\pi^2} N_f (\alpha_R - \alpha_L) \text{tr}_c(G_{\mu\nu} \tilde{G}^{\mu\nu})$ to the QCD Lagrangian, with N_f being the number of different quark flavors and $\alpha_{R/L}$ the angle of the $U(1)_{R/L}$ rotation. The vacuum angle $\theta(x)$ is in this context treated as an external pseudoscalar source that transforms under axial $U(1)$ rotations as

$$\theta(x) \rightarrow \theta'(x) = \theta(x) + i \ln \det R - \ln \det L \quad (2)$$

with $R, L \in U(1)$, so that the term in the anomaly proportional to the topological charge operator of the gluons is compensated by the shift in the θ field. There are, however, further axial anomalies which are accounted for within the effective theory by the WZW term. The dynamical variables of the effective theory are the pseudoscalar mesons (π, K, η_8, η_0) that live in the coset space $U(3)_R \times U(3)_L / U(3)_V = U(3)$. They are most conveniently collected in a unitary matrix $U(x) \in U(3)$ with a phase given by

$$\det U(x) = e^{i\psi(x)}. \quad (3)$$

The field ψ describes the singlet field η_0 and is the extension from the standard framework where the effective field is an element of $SU(3)$. Under chiral rotations, the effective field $U(x)$ transforms as

$$U'(x) = R(x)U(x)L^\dagger(x), \quad (4)$$

so that its phase changes by

$$\psi'(x) = \psi(x) - i \ln \det R + \ln \det L. \quad (5)$$

Hence, the combination $\bar{\psi} = \psi + \theta$ remains invariant in the effective theory under chiral $U(3)_R \times U(3)_L$ rotations. The covariant derivative of U involves left- and right-handed sources

$$D_\mu U = \partial_\mu U - i r_\mu U + i U l_\mu \quad (6)$$

with $r_\mu = v_\mu + a_\mu$ and $l_\mu = v_\mu - a_\mu$. In terms of these building blocks the WZW effective action is given by [3, 4]

$$\begin{aligned} S_{WZW}(U, v, a) = S_{WZW}(U) + S_{WZW}(v, a) - \frac{iN_c}{48\pi^2} \int \langle & U l^3 U^\dagger r + \frac{1}{4} U l U^\dagger r U l U^\dagger r \\ & + i U d l l U^\dagger r + i d r U l U^\dagger r - i \Sigma_L l U^\dagger r U l + \Sigma_L U^\dagger d r U l - \Sigma_L^2 U^\dagger r U l \\ & + \Sigma_L l d l + \Sigma_L d l l - i \Sigma_L l^3 + \frac{1}{2} \Sigma_L l \Sigma_L l - i \Sigma_L^3 l \rangle - (R \leftrightarrow L), \end{aligned} \quad (7)$$

where $\Sigma_L = U^\dagger dU$ and we adopted the differential form notation of [6],

$$v = dx^\mu v_\mu, \quad a = dx^\mu a_\mu, \quad r = v + a, \quad l = v - a, \quad d = dx^\mu \partial_\mu. \quad (8)$$

with the Grassmann variables dx^μ which yield the volume element $dx^\mu dx^\nu dx^\alpha dx^\beta = \epsilon^{\mu\nu\alpha\beta} d^4x$. The brackets $\langle \dots \rangle$ denote the trace in flavor space and the operation $(R \leftrightarrow L)$ indicates the interchange of r with l as well as of U with U^\dagger , so that, *e.g.*, Σ_L is replaced by $\Sigma_R = U dU^\dagger$.

In order to extract the $SU(3)$ version of the WZW term, it is convenient to introduce the notation

$$U = e^{\frac{i}{3}\bar{\psi}}\bar{U}, \quad \det \bar{U} = e^{-i\theta}. \quad (9)$$

As the field $\bar{\psi} = \psi + \theta$ is gauge invariant, \bar{U} transforms in the same manner as U under chiral rotations and its covariant derivative is defined as

$$\begin{aligned} D_\mu \bar{U} &= \partial_\mu \bar{U} - i(v_\mu + \bar{a}_\mu)\bar{U} + iU(v_\mu - \bar{a}_\mu), \\ \bar{a}_\mu &= a_\mu - \frac{1}{3}\langle a_\mu \rangle - \frac{1}{6}\partial_\mu \theta = a_\mu - \frac{1}{6}D_\mu \theta. \end{aligned} \quad (10)$$

In [6] it has been shown that the WZW term can be decomposed as

$$S_{WZW}(U, v, a) = S_{WZW}(\bar{U}, v, \bar{a}) + \int B \quad (11)$$

with

$$\begin{aligned} B &= -\frac{N_c}{144\pi^2} \left(\bar{\psi} \langle iF_{\bar{r}} D\bar{U} D\bar{U}^\dagger + iF_{\bar{l}} D\bar{U}^\dagger D\bar{U} + 2F_{\bar{r}} \bar{U} F_{\bar{l}} \bar{U}^\dagger + 2F_{\bar{r}}^2 + 2F_{\bar{l}}^2 \rangle \right. \\ &\quad \left. + \frac{1}{6} \bar{\psi} \langle F_{\bar{r}} - F_{\bar{l}} \rangle \langle F_{\bar{r}} - F_{\bar{l}} \rangle - iD\theta \langle F_{\bar{r}} D\bar{U} \bar{U}^\dagger - F_{\bar{l}} \bar{U}^\dagger D\bar{U} \rangle \right) \end{aligned} \quad (12)$$

and

$$F_{\bar{r}} = d\bar{r} - i\bar{r}^2, \quad F_{\bar{l}} = d\bar{l} - i\bar{l}^2. \quad (13)$$

The quantities \bar{r}, \bar{l} are the QCD renormalization group invariant parts of the left- and right-handed gauge fields $r = \bar{r} + \frac{1}{6}D\theta, l = \bar{l} - \frac{1}{6}D\theta$ with $D\theta = d\theta + 2\langle a \rangle$. The left- and right-hand side of Eq. (11) actually differ by two contact terms which transform in a nontrivial manner both under chiral rotations and under the QCD renormalization group. In order to obtain a renormalization group invariant anomaly, one must remove these contact terms [6]. Since these two terms involve the singlet axial vector field $\langle a_\mu \rangle$ and the derivative of the QCD vacuum angle, $\partial_\mu \theta$, they are not relevant for the present work and can safely be neglected.

The first term in Eq. (11) contains the WZW term for the $SU(3)$ effective theory

$$\int d^4x \mathcal{L}_{WZW}(\bar{U}, v, \bar{a}) \equiv S_{WZW}(\bar{U}, v, \bar{a}), \quad (14)$$

while the second one is gauge invariant and does not contribute to the anomaly. It is straightforward to show that the expression B can be absorbed by contact terms of unnatural parity at fourth chiral order

$$\begin{aligned} d^4x \tilde{\mathcal{L}}_{p^4} &= i\tilde{L}_1 \bar{\psi} \langle F_r DUDU^\dagger + F_l DU^\dagger DU \rangle + 2\tilde{L}_2 \bar{\psi} \langle F_r UF_l U^\dagger \rangle \\ &\quad + 2\tilde{L}_3 \bar{\psi} \langle F_r^2 + F_l^2 \rangle + i\tilde{L}_4 D\theta \langle F_r DUU^\dagger - F_l U^\dagger DU \rangle \\ &\quad + 2\tilde{L}_5 \bar{\psi} \left(\langle F_r \rangle \langle F_r \rangle + \langle F_l \rangle \langle F_l \rangle \right) + 2\tilde{L}_6 \bar{\psi} \langle F_r \rangle \langle F_l \rangle, \end{aligned} \quad (15)$$

where we employed the notation

$$F_r = dr - ir^2, \quad F_l = dl - il^2. \quad (16)$$

The vacuum angle θ has served its purpose and will be omitted for the rest of this section. Since we are interested in radiative decays, we will furthermore set the external vector and axial-vector fields

$$r = l = v = -eQA \quad (17)$$

with A being the photon field. The anomalous Lagrangian \mathcal{L}_{WZW} in Eq. (14) relevant for the two-photon decays at the one loop level reduces then to

$$\begin{aligned} S_{WZW}(\hat{U}, v) &= \int d^4x \mathcal{L}_{WZW}(\hat{U}, v) \\ &= -\frac{iN_c}{48\pi^2} \int d^4x \langle \hat{\Sigma}_L \hat{U}^\dagger dv \hat{U} v + \hat{\Sigma}_L v dv + \hat{\Sigma}_L dv v - i\hat{\Sigma}_L^3 v \rangle - (R \leftrightarrow L) \end{aligned} \quad (18)$$

with $U = e^{\frac{i}{3}\psi} \hat{U}$ and $\hat{\Sigma}_L = \hat{U}^\dagger d\hat{U}$.

The quark charge matrix Q of the u - d - and s -quarks has usually been assumed to be independent of the number of colors with $Q = \frac{1}{3}\text{diag}(2, -1, -1)$. However, the cancellation of triangle anomalies requires Q to depend on N_c [2]

$$Q = \text{diag}(Q_u, Q_d, Q_s) = \frac{1}{2}\text{diag}\left(\frac{1}{N_c} + 1, \frac{1}{N_c} - 1, \frac{1}{N_c} - 1\right) = \hat{Q} + \left(1 - \frac{N_c}{3}\right) \frac{1}{2N_c} \mathbf{1} \quad (19)$$

with $\hat{Q} = \frac{1}{3}\text{diag}(2, -1, -1)$ being the conventional charge matrix, while the second term is proportional to the baryon number and gives rise to the Goldstone-Wilczek term. The anomalous Lagrangian of Eq. (18) decomposes into the conventional WZW Lagrangian of the $SU(3)$ theory with the charge matrix \hat{Q} and a Goldstone-Wilczek term which vanishes for $N_c = 3$

$$S_{WZW}(\hat{U}, v) = S_{WZW}(\hat{U}, \hat{v}) + \left(1 - \frac{N_c}{3}\right) S_{GW}(\hat{U}, \hat{v}) \quad (20)$$

with $\hat{v} = -e\hat{Q}A$ and

$$\begin{aligned} S_{WZW}(\hat{U}, \hat{v}) &= \frac{N_c e}{48\pi^2} \int \langle (\hat{\Sigma}_L^3 - \hat{\Sigma}_R^3) \hat{Q} \rangle A \\ &\quad - \frac{iN_c e^2}{48\pi^2} \int \langle 2(\hat{\Sigma}_L - \hat{\Sigma}_R) \hat{Q}^2 + \hat{Q} (\hat{\Sigma}_L \hat{U}^\dagger \hat{Q} \hat{U} - \hat{\Sigma}_R \hat{U} \hat{Q} \hat{U}^\dagger) \rangle dA A, \end{aligned} \quad (21)$$

$$S_{GW}(\hat{U}, \hat{v}) = \frac{e}{48\pi^2} \int \langle \hat{\Sigma}_L^3 \rangle A - \frac{ie^2}{16\pi^2} \int \langle (\hat{\Sigma}_L - \hat{\Sigma}_R) \hat{Q} \rangle dA A. \quad (22)$$

It has been shown in [2] that the N_c dependent part of the Goldstone-Wilczek term cancels both the π - 2γ and the η - 2γ vertices of the WZW Lagrangian, yielding at tree level a decay width for these decays which does not depend on N_c , if one neglects η - η' mixing. However, at the one-loop level other vertices involving kaons will contribute to the decays. One can easily show that, *e.g.*, the vertex with two photons and π^+ , π^- , π^0 of the WZW term is canceled by the N_c dependent piece of the Goldstone-Wilczek term, in agreement with the observation that the number of colors does not appear in the effective theory for two flavors [2]. The vertices involving kaons, on the other hand, do not cancel and an N_c dependent piece remains for the vertices. Consider as an example the vertex with two photons and π^0 , K^+ , K^- that contributes

to the decay $\pi^0 \rightarrow \gamma\gamma$. The WZW term yields the vertex (neglecting mixing of the π^0 with the η - η' system)

$$-\frac{5N_c e^2}{72\pi^2} K^+ K^- d\pi^0 dAA, \quad (23)$$

whereas the N_c dependent piece of the Goldstone-Wilczek term leads to

$$\frac{N_c e^2}{36\pi^2} K^+ K^- d\pi^0 dAA. \quad (24)$$

Clearly, both terms do not compensate and a dependence on N_c remains in the final expression for the vertex. It is therefore of interest to study the N_c dependence of the two-photon decays at the one-loop level.

3 Radiative decays at one-loop order

In the framework of large N_c ChPT the expansion in powers of momenta and light quark masses is combined with the $1/N_c$ expansion by ordering the series according to

$$p = \mathcal{O}(\sqrt{\delta}), \quad m_q = \mathcal{O}(\delta), \quad 1/N_c = \mathcal{O}(\delta). \quad (25)$$

In this bookkeeping, the WZW term S_{WZW} is of order $\mathcal{O}(\delta)$, whereas the one-loop diagrams of the decays involve the ratio m_q/f^2 with $f \sim \mathcal{O}(\sqrt{\delta})$ being the pseudoscalar decay constant in the chiral limit and are thus of order $\mathcal{O}(\delta^3)$, *i.e.* next-to-next-to-leading order.

Our starting point is the WZW effective action of the $U(3)$ theory

$$\begin{aligned} S_{WZW}(U, v) &= \int d^4x \mathcal{L}_{WZW}(U, v) \\ &= -\frac{iN_c}{48\pi^2} \int d^4x \langle \Sigma_L U^\dagger dv U v + \Sigma_L v dv + \Sigma_L dv v - i\Sigma_L^3 v \rangle - (R \leftrightarrow L) \end{aligned} \quad (26)$$

We expand the quark charge matrix Q in powers of $1/N_c$

$$\begin{aligned} Q &= \frac{1}{2} \text{diag}\left(\frac{1}{N_c} + 1, \frac{1}{N_c} - 1, \frac{1}{N_c} - 1\right) = \frac{1}{2} \text{diag}(1, -1, -1) + \frac{1}{2N_c} \mathbb{1} \\ &\equiv Q^{(0)} + Q^{(1)}, \end{aligned} \quad (27)$$

where the superscript denotes the order in the combined large N_c and chiral counting scheme, *i.e.* $Q^{(0)}$ ($Q^{(1)}$) is of order $\mathcal{O}(1)$ ($\mathcal{O}(\delta)$). With $U = \exp(i\phi)$, one obtains for the three decays from S_{WZW} the tree level contributions

$$d^4x \mathcal{L}_{WZW} = \frac{N_c e^2}{8\pi^2} \langle d\phi Q^2 \rangle A dA = -\frac{N_c e^2}{8\pi^2} \langle \phi [(Q^{(0)})^2 + 2Q^{(0)}Q^{(1)} + (Q^{(1)})^2] \rangle dA dA \quad (28)$$

since $Q^{(0)}, Q^{(1)}$ commute with the diagonal entries of ϕ — ϕ^3, ϕ^8, ϕ^0 . The terms on the right-hand side of Eq. (28) contribute at orders δ, δ^2 and δ^3 , respectively, if one disregards the N_c -dependence of ϕ in which a factor $1/f = \mathcal{O}(1/\sqrt{N_c})$ has been absorbed. Hence, within large N_c ChPT the ϕ^3 and ϕ^8 decay amplitudes start at order $\mathcal{O}(\delta^2)$, whereas the decay amplitude for the singlet field ϕ^0 is of order δ .

At fourth chiral order the unnatural parity Lagrangian consists of more terms, which are gauge invariant, see Eq. (15),

$$\tilde{\mathcal{L}}_{eff} = \mathcal{L}_{WZW} + \tilde{\mathcal{L}}_{p^4} \quad (29)$$

with

$$\begin{aligned} d^4x \tilde{\mathcal{L}}_{p^4} = & 2\tilde{V}_2(\bar{\psi}) \langle F_r U F_l U^\dagger \rangle + 2\tilde{V}_3(\bar{\psi}) \langle F_r^2 + F_l^2 \rangle \\ & + 2\tilde{V}_5(\bar{\psi}) \left(\langle F_r \rangle \langle F_r \rangle + \langle F_l \rangle \langle F_l \rangle \right) + 2\tilde{V}_6(\bar{\psi}) \langle F_r \rangle \langle F_l \rangle + \dots, \end{aligned} \quad (30)$$

where we have presented only the contact terms which contribute to the decays at the order we are working.

The potentials \tilde{V}_i are odd functions in $\bar{\psi}$, so that the leading contribution in the $1/N_c$ expansion is linear in $\bar{\psi}$. The Lagrangian $\tilde{\mathcal{L}}_{p^4}$ can be expanded in powers of $1/N_c$

$$\tilde{\mathcal{L}}_{p^4} = \tilde{\mathcal{L}}_{p^4}^{(2)} + \tilde{\mathcal{L}}_{p^4}^{(3)} + \dots, \quad (31)$$

where the superscript denotes the order in the δ expansion with

$$d^4x \tilde{\mathcal{L}}_{p^4}^{(2)} = 2\tilde{L}_2 \bar{\psi} \langle F_r U F_l U^\dagger \rangle + 2\tilde{L}_3 \bar{\psi} \langle F_r^2 + F_l^2 \rangle \quad (32)$$

and

$$d^4x \tilde{\mathcal{L}}_{p^4}^{(3)} = 2\tilde{L}_5 \bar{\psi} \left(\langle F_r \rangle \langle F_r \rangle + \langle F_l \rangle \langle F_l \rangle \right) + 2\tilde{L}_6 \bar{\psi} \langle F_r \rangle \langle F_l \rangle. \quad (33)$$

The contributions from $\tilde{\mathcal{L}}_{p^4}^{(2)}$ and $\tilde{\mathcal{L}}_{p^4}^{(3)}$ are of order $\mathcal{O}(p^4)$ and $\mathcal{O}(\frac{1}{N_c}p^4)$, respectively. Setting $F_r = F_l = -eQ dA$, we obtain for $\tilde{\mathcal{L}}_{p^4}^{(2)}$

$$\begin{aligned} d^4x \tilde{\mathcal{L}}_{p^4}^{(2)} &= 2e^2 [\tilde{L}_2 + 2\tilde{L}_3] \bar{\psi} \langle (Q^{(0)})^2 + 2Q^{(0)}Q^{(1)} + (Q^{(1)})^2 \rangle dA dA \\ &= e^2 k_1 \left(\frac{3}{4} - \frac{1}{2N_c} \right) \bar{\psi} dA dA \end{aligned} \quad (34)$$

where $k_1 = 2(\tilde{L}_2 + 2\tilde{L}_3)$ and the last term proportional to $(Q^{(1)})^2$ has been omitted, since it is of order $\mathcal{O}(\delta^4)$ and thus beyond our working precision. In a similar way, the terms from $\tilde{\mathcal{L}}_{p^4}^{(3)}$ reduce to

$$d^4x \tilde{\mathcal{L}}_{p^4}^{(3)} = e^2 k_2 \bar{\psi} \langle Q^{(0)} \rangle \langle Q^{(0)} \rangle dA dA \quad (35)$$

with $k_2 = 2(2\tilde{L}_5 + \tilde{L}_6)$.

From the renormalization group invariance of the effective Lagrangian it follows that k_1 and k_2 transform as (*cf.* App. A for details)

$$\begin{aligned} k_1^{ren} &= Z_A k_1 - \frac{N_c(Z_A - 1)}{24\pi^2} \\ k_2^{ren} &= Z_A k_2, \end{aligned} \quad (36)$$

where Z_A is the multiplicative renormalization constant of the singlet axial current $A_\mu^0 = \frac{1}{2}\bar{q}\gamma_\mu\gamma_5 q$ which transforms as $(A_\mu^0)^{ren} = Z_A A_\mu^0$ under changes in the QCD running scale.

At sixth chiral order the relevant terms for the decays read

$$\begin{aligned} d^4x \tilde{\mathcal{L}}_\chi = & i\tilde{W}_1(\bar{\psi}) \langle U \chi^\dagger F_r^2 + \chi^\dagger U F_l^2 \rangle + i\tilde{W}_2(\bar{\psi}) \langle \chi^\dagger F_r U F_l + U \chi^\dagger U F_l U^\dagger F_r \rangle \\ & + i\tilde{W}_3(\bar{\psi}) \langle U \chi^\dagger \rangle \langle F_r^2 + F_l^2 \rangle + i\tilde{W}_4(\bar{\psi}) \langle U \chi^\dagger \rangle \langle U^\dagger F_r U F_l \rangle \\ & + i\tilde{W}_5(\bar{\psi}) \langle F_r + F_l \rangle \langle [F_r U + U F_l] \chi^\dagger \rangle + h.c. \quad . \end{aligned} \quad (37)$$

The quark mass matrix $\mathcal{M} = \text{diag}(m_u, m_d, m_s)$ enters in the combination $\chi = 2B\mathcal{M}$ with $B = -\langle 0|\bar{q}q|0\rangle/f^2$ being the order parameter of the spontaneous symmetry violation. Expanding in powers of $1/N_c$ one obtains

$$\tilde{\mathcal{L}}_\chi = \tilde{\mathcal{L}}_\chi^{(2)} + \tilde{\mathcal{L}}_\chi^{(3)} \quad (38)$$

with the contributions $\tilde{\mathcal{L}}_\chi^{(2)}$ at $\mathcal{O}(N_c p^6)$

$$\begin{aligned} d^4x \tilde{\mathcal{L}}_\chi^{(2)} = & i\tilde{w}_1^{(0)} \langle [U\chi^\dagger - \chi U^\dagger] F_r^2 + [\chi^\dagger U - U^\dagger \chi] F_l^2 \rangle \\ & + i\tilde{w}_2^{(0)} \langle [\chi^\dagger - U^\dagger \chi U^\dagger] F_r U F_l + [U\chi^\dagger U - \chi] F_l U^\dagger F_r \rangle \end{aligned} \quad (39)$$

and $\tilde{\mathcal{L}}_\chi^{(3)}$ at order $\mathcal{O}(p^6)$

$$\begin{aligned} d^4x \tilde{\mathcal{L}}_\chi^{(3)} = & -\tilde{w}_1^{(1)} \bar{\psi} \langle [U\chi^\dagger + \chi U^\dagger] F_r^2 + [\chi^\dagger U + U^\dagger \chi] F_l^2 \rangle \\ & - \tilde{w}_2^{(1)} \bar{\psi} \langle [\chi^\dagger + U^\dagger \chi U^\dagger] F_r U F_l + [U\chi^\dagger U + \chi] F_l U^\dagger F_r \rangle \\ & + i\tilde{w}_3^{(0)} \langle U\chi^\dagger - \chi U^\dagger \rangle \langle F_r^2 + F_l^2 \rangle + i\tilde{w}_4^{(0)} \langle U\chi^\dagger - \chi U^\dagger \rangle \langle U^\dagger F_r U F_l \rangle \\ & + i\tilde{w}_5^{(0)} \langle F_r + F_l \rangle \langle F_r U \chi^\dagger - U^\dagger F_r \chi + U F_l \chi^\dagger - F_l U^\dagger \chi \rangle, \end{aligned} \quad (40)$$

where the potentials \tilde{W}_i have been expanded according to $\tilde{W}_i = \tilde{w}_i^{(0)} + i\tilde{w}_i^{(1)}\bar{\psi} + \mathcal{O}(\bar{\psi}^2)$.

The explicitly symmetry breaking terms reduce to the structures

$$d^4x \tilde{\mathcal{L}}_\chi^{(2)} = k_3 e^2 \langle \phi \chi [(Q^{(0)})^2 + 2Q^{(0)}Q^{(1)}] \rangle dA dA \quad (41)$$

with $k_3 = -4(\tilde{w}_1^{(0)} + \tilde{w}_2^{(0)})$ and

$$d^4x \tilde{\mathcal{L}}_\chi^{(3)} = e^2 \left(k_4 \bar{\psi} \langle \chi (Q^{(0)})^2 \rangle + k_5 \langle \phi \chi \rangle \langle (Q^{(0)})^2 \rangle + k_6 \langle Q^{(0)} \rangle \langle \phi \chi Q^{(0)} \rangle \right) dA dA \quad (42)$$

with $k_4 = -4(\tilde{w}_1^{(1)} + \tilde{w}_2^{(1)})$, $k_5 = -2(2\tilde{w}_3^{(0)} + \tilde{w}_4^{(0)})$, $k_6 = -8\tilde{w}_5^{(0)}$, respectively. The scaling law for the parameter k_4 is given by (*cf.* App. A)

$$k_4^{ren} = Z_A k_4 + \frac{1}{3}[Z_A - 1]k_3, \quad (43)$$

while the remaining parameters k_3, k_5 and k_6 remain put.

Having discussed the tree diagram contributions to the decays, we now turn to the calculation of the loops at order δ^3 . After expanding the WZW Lagrangian in the meson fields ϕ the contributing pieces at one-loop order read

$$\begin{aligned} d^4x \mathcal{L}_{wzw} = & -\frac{N_c e^2}{48\pi^2} \left(\langle d\phi[\phi, [\phi, Q^2]] \rangle - \langle d\phi[\phi, Q][\phi, Q] \rangle \right) A dA \\ & - \frac{iN_c e}{24\pi^2} \langle d\phi d\phi d\phi Q \rangle A + \dots \end{aligned} \quad (44)$$

The mesons inside the loops do not undergo mixing, as ϕ^3, ϕ^8 and ϕ^0 loops do not contribute. The first two terms in Eq. (44) contribute via tadpoles, whereas the last one represents a vertex of the unitarity correction, corresponding to Figure 1b. As the diagonal components of ϕ commute with Q , tadpoles with ϕ^3, ϕ^8 and ϕ^0 do not contribute, and since the photon couples only to charged mesons, the unitarity corrections arise due to charged meson loops.

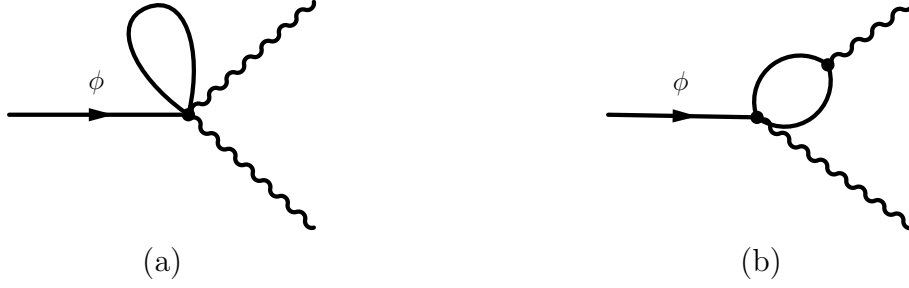


Figure 1: One-loop diagrams contributing to $\phi \rightarrow \gamma\gamma$. In (b) the crossed diagram is not shown.

At order $\mathcal{O}(\delta^3)$ only the $Q^{(0)}$ piece in Eq. (44) contributes. From $(Q^{(0)})^2 = \frac{1}{4}\mathbb{1}$ it follows then that the first term vanishes and the tadpoles are entirely due to the second term. Performing the loop integration, one obtains for the decays the tadpole contributions

$$\frac{N_c e^2}{24\pi^2 f^2} \sum_{i=3,8,0} \left(d_\pi^i \Delta_\pi + d_K^i \Delta_K \right) \phi^i dA dA \quad (45)$$

with the tadpole given in dimensional regularization by

$$\Delta_\phi = \int \frac{d^d l}{(2\pi)^d} \frac{i}{l^2 - m_\phi^2 + i\epsilon} = m_\phi^2 \left[2L + \frac{1}{16\pi^2} \ln \frac{m_\phi^2}{\mu^2} \right], \quad (46)$$

where L contains the pole at $d = 4$,

$$L = \frac{\mu^{d-4}}{16\pi^2} \left(\frac{1}{d-4} - \frac{1}{2} (\ln 4\pi + \Gamma'(1) + 1) \right) \quad (47)$$

and μ is the regularization scale. The coefficients d_ϕ^i read

$$\begin{aligned} d_\pi^3 &= 0, & d_\pi^8 &= -\sqrt{\frac{2}{3}}, & d_\pi^0 &= -\frac{2}{\sqrt{3}}; \\ d_K^3 &= -\frac{1}{\sqrt{2}}, & d_K^8 &= \frac{1}{\sqrt{6}}, & d_K^0 &= -\frac{2}{\sqrt{3}}. \end{aligned} \quad (48)$$

Evaluating the unitarity corrections at order $\mathcal{O}(\delta^3)$ for on-shell photons and replacing Q in Eq. (44) by $Q^{(0)}$, since the contribution from $Q^{(1)}$ is beyond the order we are working, one obtains exactly the same contribution as for the tadpoles but with opposite sign. Hence, the one-loop corrections to the decays at order $\mathcal{O}(\delta^3)$ compensate each other and the first non-vanishing non-analytic piece will show up at order $\mathcal{O}(\delta^4)$. This is also in agreement with previous calculations in conventional ChPT, in which the chiral logarithms were compensated completely by wavefunction renormalization and replacing f by the physical decay constant F_ϕ in the tree level expression [10, 11]. However, within large N_c ChPT the ϕ^3 and ϕ^8 decay amplitudes start at order $\mathcal{O}(\delta^2)$, so that the leading non-analytic corrections to the physical decay constant and wavefunction renormalization will contribute at order $\mathcal{O}(\delta^4)$ and do not affect the amplitude up to order $\mathcal{O}(\delta^3)$. Any divergences from the loop diagrams discussed above could then only be renormalized by counter terms of the p^6 Lagrangian of unnatural parity. This would be clearly in contradiction to previous results [12, 13].

So far, the ϕ^8 amplitude does not contain an explicit N_c dependence due to the WZW term, but both the ϕ^8 and ϕ^0 fields undergo mixing which results in the mass eigenstates η and η' . Here, we work in the isospin limit of equal up- and down-quark masses, $\hat{m} = m_u = m_d$, so that the ϕ^3 field decouples from the η - η' system. In the next section, we will give an estimate on isospin breaking effects due to different quark masses $m_u \neq m_d$.

In order to describe η - η' mixing up to one-loop order, one must take into account the following terms of the effective Lagrangian

$$\mathcal{L}_{eff} = \mathcal{L}^{(0)} + \mathcal{L}^{(1)} + \mathcal{L}^{(2)} + \dots \quad (49)$$

which reads at lowest order δ^0

$$\mathcal{L}^{(0)} = \frac{f^2}{4} \langle D_\mu U^\dagger D^\mu U \rangle + \frac{f^2}{4} \langle \chi U^\dagger + U \chi^\dagger \rangle - \frac{1}{2} \tau \bar{\psi}^2. \quad (50)$$

The next-to-leading order Lagrangian $\mathcal{L}^{(1)} = \mathcal{O}(\delta)$ contains the terms

$$\begin{aligned} \mathcal{L}^{(1)} = & L_5 \langle D_\mu U^\dagger D^\mu U (\chi^\dagger U + U^\dagger \chi) \rangle + L_8 \langle \chi^\dagger U \chi^\dagger U + U^\dagger \chi U^\dagger \chi \rangle \\ & + \frac{f^2}{12} \Lambda_1 D_\mu \psi D^\mu \psi + i \frac{f^2}{12} \Lambda_2 \bar{\psi} \langle \chi^\dagger U - U^\dagger \chi \rangle, \end{aligned} \quad (51)$$

and $\mathcal{L}^{(2)}$ is given by

$$\begin{aligned} \mathcal{L}^{(2)} = & L_4 \langle D_\mu U^\dagger D^\mu U \rangle \langle \chi^\dagger U + U^\dagger \chi \rangle + L_6 \langle \chi^\dagger U + U^\dagger \chi \rangle^2 + L_7 \langle \chi^\dagger U - U^\dagger \chi \rangle^2 \\ & + i L_{18} D_\mu \psi \langle D^\mu U^\dagger \chi - D^\mu U \chi^\dagger \rangle + i L_{25} \bar{\psi} \langle U^\dagger \chi U^\dagger \chi - \chi^\dagger U \chi^\dagger U \rangle + \mathcal{O}(N_c p^6). \end{aligned} \quad (52)$$

The terms $\eta_0^2 \langle D_\mu U^\dagger D^\mu U \rangle$ and $\eta_0^2 \langle \chi U^\dagger + U \chi^\dagger \rangle$ have been omitted in $\mathcal{L}^{(2)}$, since the pertinent unknown coupling constants represent OZI violating corrections. Moreover, as indicated in the last equation, counterterms of order $\mathcal{O}(N_c p^6)$ with new unknown coupling constants will contribute at order δ^2 . We will neglect these contributions throughout, assuming that they are of small size and do not alter our conclusions.

The fields ϕ^8 and ϕ^0 are related to the mass eigenstates η and η' via

$$\begin{aligned} \phi^8 = & \frac{\sqrt{2}}{F_\eta^8} [\cos \vartheta^{(1)} - \sin \vartheta^{(0)} \mathcal{A}^{(1)}] \eta + \frac{\sqrt{2}}{F_\eta^8} [\sin \vartheta^{(1)} + \cos \vartheta^{(0)} \mathcal{A}^{(1)}] \eta' \\ \phi^0 = & \frac{\sqrt{2}}{\sqrt{3} F_{\eta'}^0} [\cos \vartheta^{(1)} \mathcal{A}^{(2)} - \sin \vartheta^{(2)} \mathcal{B}] \eta + \frac{\sqrt{2}}{\sqrt{3} F_{\eta'}^0} [\sin \vartheta^{(1)} \mathcal{A}^{(2)} + \cos \vartheta^{(2)} \mathcal{B}] \eta'. \end{aligned} \quad (53)$$

The decay constants F_η^8 and $F_{\eta'}^0$ are defined by

$$\begin{aligned} \langle 0 | \bar{q} \gamma_\mu \gamma_5 \lambda^8 q | \eta \rangle &= i \sqrt{2} p_\mu F_\eta^8 \\ \langle 0 | \bar{q} \gamma_\mu \gamma_5 \lambda^0 q | \eta' \rangle &= i \sqrt{2} p_\mu F_{\eta'}^0 \end{aligned} \quad (54)$$

with the normalization $\langle \lambda^a \lambda^b \rangle = \delta^{ab}$, while the angles $\vartheta^{(i)}$ correspond to the mixing angle up

to order $\mathcal{O}(\delta^i)$ which arises in the diagonalization of the ϕ^8 - ϕ^0 mass matrix

$$\begin{aligned}
\sin 2\vartheta^{(0)} &= -\frac{4\sqrt{2}m_K^2 - m_\pi^2}{3(m_{\eta'}^2 - m_\eta^2)} \\
\sin 2\vartheta^{(1)} &= \sin 2\vartheta^{(0)} \left(\frac{1 + \Lambda_2}{\sqrt{1 + \Lambda_1}} + \frac{8}{F_\pi^2} [2L_8^{(r)} - L_5^{(r)}] (m_K^2 - m_\pi^2) - \frac{24}{F_\pi^4} L_5^{(r)} \tau \right) \\
\sin 2\vartheta^{(2)} &= \frac{2\sqrt{2}}{3[m_{\eta'}^2 - m_\eta^2]} \left(2[\overset{\circ}{m}_\pi^2 - \overset{\circ}{m}_K^2] \frac{1 + \Lambda_2}{\sqrt{1 + \Lambda_1}} + \frac{32}{F_\pi^2} [m_\pi^2 - m_K^2] m_K^2 \frac{2L_8^{(r)} - 3L_{25}}{\sqrt{1 + \Lambda_1}} \right. \\
&\quad + \frac{16}{F_\pi^2} [m_\pi^2 - m_K^2] [2m_K^2 + m_\pi^2] (2L_6^{(r)} + 2L_7 - L_4^{(r)}) \\
&\quad - \frac{24}{F_\pi^4} [m_\pi^2 - m_K^2] (2L_5^{(r)} + 3L_{18}^{(r)}) \tau (1 - \frac{5}{4}\Lambda_1) + \frac{64}{F_\pi^6} [m_\pi^2 - m_K^2] [11m_K^2 + m_\pi^2] L_5^{(r)2} \tau \\
&\quad - \frac{16}{F_\pi^2} [m_\pi^2 - m_K^2] [7m_K^2 + m_\pi^2] L_5^{(r)} (1 - \frac{1}{4}\Lambda_1) \\
&\quad + \frac{8}{F_\pi^2} [m_\pi^2 - m_K^2] [5m_K^2 + m_\pi^2] (2L_5^{(r)} (1 - \frac{1}{3}\Lambda_2) - L_{18}^{(r)}) - \frac{3}{2F_\pi^2} m_\pi^2 \Delta_\pi^{(r)} + \frac{1}{F_\pi^2} m_K^2 \Delta_K^{(r)} \\
&\quad - \frac{1}{3F_\pi^2} \Delta_\eta^{(r)} (m_\pi^2 [\frac{5}{2} \cos^2 \vartheta^{(0)} + \frac{1}{\sqrt{2}} \sin 2\vartheta^{(0)} + 2 \sin^2 \vartheta^{(0)}] \\
&\quad \quad \quad - m_K^2 [4 \cos^2 \vartheta^{(0)} + 2\sqrt{2} \sin 2\vartheta^{(0)} + 2 \sin^2 \vartheta^{(0)}]) \\
&\quad - \frac{1}{3F_\pi^2} \Delta_{\eta'}^{(r)} (m_\pi^2 [\frac{5}{2} \sin^2 \vartheta^{(0)} - \frac{1}{\sqrt{2}} \sin 2\vartheta^{(0)} + 2 \cos^2 \vartheta^{(0)}] \\
&\quad \quad \quad - m_K^2 [4 \sin^2 \vartheta^{(0)} - 2\sqrt{2} \sin 2\vartheta^{(0)} + 2 \cos^2 \vartheta^{(0)}]) + \mathcal{O}(N_c p^6) \Big) \quad (55)
\end{aligned}$$

where $F_\pi \approx 93$ MeV is the pion decay constant defined in a similar way as in Eq. (54), and $m_\eta, m_{\eta'}$ are the diagonal entries of the η - η' mass matrix. Furthermore, $L_i^{(r)}$ and $\Delta_\phi^{(r)} = m_\phi^2/(16\pi^2) \ln(m_\phi^2/\mu^2)$ are the finite parts of the LECs and the loops, respectively, after renormalization. It is straightforward to verify that the angles $\vartheta^{(i)}$ do not depend on the regularization scale of the effective theory.

The quantities $\overset{\circ}{m}_\pi$ and $\overset{\circ}{m}_K$ denote the pion and kaon masses at leading order

$$\begin{aligned}
\overset{\circ}{m}_\pi^2 &= 2B\hat{m} = m_\pi^2 \left(1 - \frac{8}{F_\pi^2} m_\pi^2 [2L_8^{(r)} - L_5^{(r)}] - \frac{8}{F_\pi^2} (2m_K^2 + m_\pi^2) [2L_6^{(r)} - L_4^{(r)}] - \frac{1}{2F_\pi^2} \Delta_\pi^{(r)} \right. \\
&\quad \left. + \frac{1}{6F_\pi^2} [\cos^2 \vartheta^{(0)} + 2 \sin^2 \vartheta^{(0)}] \Delta_\eta^{(r)} + \frac{1}{6F_\pi^2} [\sin^2 \vartheta^{(0)} + 2 \cos^2 \vartheta^{(0)}] \Delta_{\eta'}^{(r)} \right), \\
\overset{\circ}{m}_K^2 &= B(\hat{m} + m_s) = m_K^2 \left(1 - \frac{8}{F_\pi^2} m_K^2 [2L_8^{(r)} - L_5^{(r)}] - \frac{8}{F_\pi^2} (2m_K^2 + m_\pi^2) [2L_6^{(r)} - L_4^{(r)}] \right. \\
&\quad \left. - \frac{1}{3F_\pi^2} \cos 2\vartheta^{(0)} \Delta_\eta^{(r)} \right) + \frac{1}{12F_\pi^2} [4m_K^2 \cos^2 \vartheta^{(0)} - (3m_{\eta'}^2 + m_\pi^2) \sin^2 \vartheta^{(0)}] \Delta_{\eta'}^{(r)}. \quad (56)
\end{aligned}$$

The expressions $\mathcal{A}^{(i)}$ and \mathcal{B} read

$$\begin{aligned}
\mathcal{A}^{(1)} &= \frac{8\sqrt{2}}{3F_\pi^2} L_5^{(r)} [m_K^2 - m_\pi^2] \\
\mathcal{A}^{(2)} &= \frac{4\sqrt{2}}{3F_\pi^2} [m_K^2 - m_\pi^2] \left(\frac{2L_5^{(r)} + 3L_{18}^{(r)}}{(1 + \Lambda_1)^{1/4}} + \frac{8}{3F_\pi^2} L_5^{(r)2} [-m_K^2 + 13m_\pi^2] - \frac{32}{F_\pi^2} L_5^{(r)} L_8^{(r)} [m_K^2 + m_\pi^2] \right) \\
\mathcal{B} &= 1 + \frac{4}{3F_\pi^2} [2m_K^2 + m_\pi^2] \left(3L_4^{(r)} - L_5^{(r)} + \frac{2L_5^{(r)} + 3L_{18}^{(r)}}{\sqrt{1 + \Lambda_1}} - \frac{3L_4^{(r)} + L_5^{(r)} + 3L_{18}^{(r)}}{1 + \Lambda_1} \right) \\
&\quad + \frac{64}{9F_\pi^4} L_5^{(r)2} [3m_K^4 - 4m_\pi^2 m_K^2 + 3m_\pi^4]. \tag{57}
\end{aligned}$$

For the ϕ^8 decay we have only kept the pieces up to next-to-leading order, since the terms beyond that order contribute at $\mathcal{O}(\delta^4)$. In the case of the ϕ^0 decay, on the other hand, one must keep also the contributions at next-to-next-to-leading order. The values of the couplings $\Lambda_1, \Lambda_2, L_{18}^{(r)}$ and L_{25} are not known and depend on the running scale of QCD. As they represent OZI violating corrections, we will omit them, but, strictly speaking, we cannot expect that all neglected terms vanish at the same scale. Furthermore, the parameter τ is related to the mass of the η' in the chiral limit which was estimated in [9] to be about 850 MeV. This translates into a value of $\tau \approx 1 \times 10^{-3} \text{ GeV}^4$.

Including the mixing from Eq. (55) we obtain the amplitudes

$$\begin{aligned}
e^2 \left[\frac{B_\pi}{F_\pi} \pi^0 + \left(\frac{B_\eta}{F_\eta^8} [\cos \vartheta^{(1)} - \sin \vartheta^{(0)} \mathcal{A}^{(1)}] + \frac{B_{\eta'}}{F_{\eta'}^0} [\cos \vartheta^{(1)} \mathcal{A}^{(2)} - \sin \vartheta^{(2)} \mathcal{B}] \right) \eta \right. \\
\left. + \left(\frac{B_\eta}{F_\eta^8} [\sin \vartheta^{(1)} + \cos \vartheta^{(0)} \mathcal{A}^{(1)}] + \frac{B_{\eta'}}{F_{\eta'}^0} [\sin \vartheta^{(1)} \mathcal{A}^{(2)} + \cos \vartheta^{(2)} \mathcal{B}] \right) \eta' \right] dA \, dA, \tag{58}
\end{aligned}$$

where π^0 is related to the ϕ^3 field via

$$\pi^0 = \frac{f}{\sqrt{2Z_\pi}} \phi^3 = \frac{F_\pi}{\sqrt{2}} \phi^3 \tag{59}$$

with the pion Z -factor

$$\sqrt{Z_\pi} = 1 - \frac{4}{f^2} m_\pi^2 L_5^{(r)} \tag{60}$$

and the decay constant

$$F_\pi = f \left(1 + \frac{4}{f^2} m_\pi^2 L_5^{(r)} \right) \tag{61}$$

to the order we are working. The coefficients B_ϕ read

$$\begin{aligned}
B_\pi &= -\frac{N_c}{4\sqrt{2}\pi^2} \langle \lambda_3 Q^2 \rangle + \sqrt{2}k_3 \langle \lambda_3 \chi [(Q^{(0)})^2 + 2Q^{(0)}Q^{(1)}] \rangle \\
&\quad + \sqrt{2}k_5 \langle \lambda_3 \chi \rangle \langle (Q^{(0)})^2 \rangle + \sqrt{2}k_6 \langle Q^{(0)} \rangle \langle \lambda_3 \chi Q^{(0)} \rangle \\
B_\eta &= -\frac{N_c}{4\sqrt{2}\pi^2} \langle \lambda_8 Q^2 \rangle + \sqrt{2}k_3 \langle \lambda_8 \chi [(Q^{(0)})^2 + 2Q^{(0)}Q^{(1)}] \rangle \\
&\quad + \sqrt{2}k_5 \langle \lambda_8 \chi \rangle \langle (Q^{(0)})^2 \rangle + \sqrt{2}k_6 \langle Q^{(0)} \rangle \langle \lambda_8 \chi Q^{(0)} \rangle \\
B_{\eta'} &= -\frac{N_c}{4\sqrt{6}\pi^2} \langle Q^2 \rangle + \sqrt{6}k_1 \langle (Q^{(0)})^2 + 2Q^{(0)}Q^{(1)} \rangle + \sqrt{6}k_2 \langle Q^{(0)} \rangle \langle Q^{(0)} \rangle \\
&\quad + \sqrt{\frac{2}{3}}k_3 \langle \chi [(Q^{(0)})^2 + 2Q^{(0)}Q^{(1)}] \rangle + \sqrt{6}k_4 \langle \chi (Q^{(0)})^2 \rangle \\
&\quad + \sqrt{\frac{2}{3}}k_5 \langle \chi \rangle \langle (Q^{(0)})^2 \rangle + \sqrt{\frac{2}{3}}k_6 \langle Q^{(0)} \rangle \langle \chi Q^{(0)} \rangle.
\end{aligned} \tag{62}$$

Due to the N_c dependence of the quark charge matrix Q the expressions B_π and B_η start at order δ^2 , whereas $B_{\eta'}$ contains a piece of order $\mathcal{O}(\delta)$. Substituting these relations into Eq. (58) yields the decay widths

$$\begin{aligned}
\Gamma_{\pi^0 \rightarrow \gamma\gamma} &= \alpha^2 \pi m_{\pi^0}^3 \left| \frac{B_\pi}{F_\pi} \right|^2, \\
\Gamma_{\eta \rightarrow \gamma\gamma} &= \alpha^2 \pi m_\eta^3 \left| \frac{B_\eta}{F_\eta^8} [\cos \vartheta^{(1)} - \sin \vartheta^{(0)} \mathcal{A}^{(1)}] + \frac{B_{\eta'}}{F_{\eta'}^0} [\cos \vartheta^{(1)} \mathcal{A}^{(2)} - \sin \vartheta^{(2)} \mathcal{B}] \right|^2, \\
\Gamma_{\eta' \rightarrow \gamma\gamma} &= \alpha^2 \pi m_{\eta'}^3 \left| \frac{B_\eta}{F_\eta^8} [\sin \vartheta^{(1)} + \cos \vartheta^{(0)} \mathcal{A}^{(1)}] + \frac{B_{\eta'}}{F_{\eta'}^0} [\sin \vartheta^{(1)} \mathcal{A}^{(2)} + \cos \vartheta^{(2)} \mathcal{B}] \right|^2
\end{aligned} \tag{63}$$

with $\alpha = e^2/4\pi$. In the δ expansion the leading order contribution to the decay width of the η is given due to mixing by the leading contribution in $B_{\eta'}$, and is comparable in size with the B_η portion. The numerical values will be discussed in detail in the next section.

4 Numerical analysis

Equations (55) and (57) are utilized to obtain values for the mixing angles $\vartheta^{(i)}$ and the expressions $\mathcal{A}^{(i)}$, \mathcal{B} , respectively. For the LECs $L_{4,5,6,8}^{(r)}$, L_7 , we take values which follow from matching the $U(3)$ theory to the $SU(3)$ framework by integrating out the singlet field [6, 19] at the regularization scale $\mu = 1$ GeV, $L_4^{(r)}(\mu) = -0.5$, $L_5^{(r)}(\mu) = 1.0$, $L_6^{(r)}(\mu) = -0.3$, $L_7 = -0.3$, $L_8^{(r)}(\mu) = 0.7$ (all in units of 10^{-3}). Note that integrating out the singlet field only alters the LECs $L_6^{(r)}$, L_7 and $L_8^{(r)}$ and their values in the $U(3)$ framework are within the phenomenologically determined error ranges of the $SU(3)$ LECs. Moderate variations of these LECs yield small changes in the decay amplitudes with the largest changes induced by variations in L_7 roughly at the 10% level. Our conclusions are therefore not altered, if slightly different values for the LECs are employed. Using the experimental values for the pseudoscalar meson masses, we obtain $\vartheta^{(0)} = -21.8^\circ$, $\vartheta^{(1)} = -15.8^\circ$, $\vartheta^{(2)} = -19.8^\circ$.

By employing Eq. (63), we can now fit the ratios B_π/F_π , B_η/F_η^8 , $B_{\eta'}/F_{\eta'}^0$ to the decay widths $\Gamma_{\pi^0 \rightarrow \gamma\gamma}$, $\Gamma_{\eta \rightarrow \gamma\gamma}$, $\Gamma_{\eta' \rightarrow \gamma\gamma}$. The experimental values for the decay widths are [20]

$$\begin{aligned}\Gamma_{\pi^0 \rightarrow \gamma\gamma} &= 7.74 \pm 0.55 \text{ eV}, \\ \Gamma_{\eta \rightarrow \gamma\gamma} &= 0.465 \pm 0.045 \text{ keV}, \\ \Gamma_{\eta' \rightarrow \gamma\gamma} &= 4.28 \pm 0.34 \text{ keV},\end{aligned}\tag{64}$$

and the fit to the central values yields

$$\begin{aligned}B_\pi/F_\pi &= -0.133 \text{ GeV}^{-1}, \\ B_\eta/F_\eta^8 &= -0.0522 \text{ GeV}^{-1}, \\ B_{\eta'}/F_{\eta'}^0 &= -0.192 \text{ GeV}^{-1}.\end{aligned}\tag{65}$$

For the pion decay constant F_π we employ the physical value $F_\pi \approx 93 \text{ MeV}$, while F_η^8 can be extracted from a one-loop calculation with $F_\eta^8 = 1.34 F_\pi$ [7]. It is consistent to take the one-loop results for F_π and F_η^8 , since the difference with respect to the next-to-leading order expressions shows up at δ^4 in the decay amplitude and is, therefore, beyond our working precision. The values for B_π/F_π and B_η/F_η^8 from the fit are close to the contributions from the anomalous WZW term, $B_\pi^{WZW}/F_\pi = -0.136 \text{ GeV}^{-1}$, $B_\eta^{WZW}/F_\eta^8 = -0.0587 \text{ GeV}^{-1}$, indicating that the portions from the counter terms of unnatural parity are small. Omitting higher orders beyond δ^3 , they contribute with a relative strength of about 2% to B_π and 10% to B_η . In order to get an estimate for $F_{\eta'}^0$, it is thus justified to assume that the counter term combination in $B_{\eta'}$ is small as well. Of course, both the counter terms and $F_{\eta'}^0$ depend on the renormalization scale μ_{QCD} , but we will assume that for a certain range of μ_{QCD} the counter term contributions are negligible. For such μ_{QCD} , the ratio $B_{\eta'}^{WZW}/F_{\eta'}^0$ is then reproduced by setting $F_{\eta'}^0 \approx 1.16 F_\pi$, a value slightly larger than in previous calculations [10, 11].

In particular, we would like to investigate, whether a clear statement can be given on the number of colors by utilizing the $1/N_c$ expansions of the decay amplitudes. The cancellation of Witten's global $SU(2)_L$ anomaly requires N_c to be odd [22]. The standard model with $N_c = 1$ is without strong interactions. We will therefore restrict ourselves to a comparison of the numerical results for $N_c = 3$ and $N_c = 5$. Setting all non-anomalous contact terms of unnatural parity to zero, the decay width for the η in a world with $N_c = 5$ reads $\Gamma_{\eta \rightarrow \gamma\gamma}^{N_c=5} = 1.002 \text{ keV}$ to be compared with the decay width in the real world with three colors, $\Gamma_{\eta \rightarrow \gamma\gamma}^{N_c=3} = 0.511 \text{ keV}$. For the η' we obtain $\Gamma_{\eta' \rightarrow \gamma\gamma}^{N_c=3} = 4.21 \text{ keV}$ and $\Gamma_{\eta' \rightarrow \gamma\gamma}^{N_c=5} = 12.8 \text{ keV}$. The experimental values for the η and η' decays clearly rule out $N_c = 5$ and varying the values for the omitted coupling constants of the counter terms within realistic ranges does not alter this conclusion.

Finally, we would like to give an estimate on the N_c dependence of the π^0 decay width due to different up- and down-quark masses. In the case of different up- and down-quark masses, the ϕ^3 field undergoes mixing with both the ϕ^8 and ϕ^0 field. In order to get an estimate, we will restrict ourselves to the mixing at leading order in the δ expansion. The fields ϕ^3 , ϕ^8 and

ϕ^0 are then related to the mass eigenstates via

$$\begin{aligned}\phi^3 &= \frac{\sqrt{2}}{F_\pi} (\pi^0 - \epsilon\eta - \epsilon'\eta') \\ \phi^8 &= \frac{\sqrt{2}}{F_\eta^8} (\cos\vartheta^{(0)} (\eta + \epsilon\pi^0) + \sin\vartheta^{(0)} (\eta' + \epsilon'\pi^0)) \\ \phi^0 &= \frac{\sqrt{2}}{\sqrt{3}F_{\eta'}^0} (-\sin\vartheta^{(0)} (\eta + \epsilon\pi^0) + \cos\vartheta^{(0)} (\eta' + \epsilon'\pi^0))\end{aligned}\quad (66)$$

with the mixing parameters

$$\begin{aligned}\epsilon_0 &= \frac{\sqrt{3}}{4} \frac{m_d - m_u}{m_s - \hat{m}}, \\ \epsilon &= \epsilon_0 \frac{\cos\vartheta^{(0)} - \sqrt{2}\sin\vartheta^{(0)}}{1 + \frac{1}{\sqrt{2}}\tan\vartheta^{(0)}}, \\ \epsilon' &= \epsilon_0 \frac{\sin\vartheta^{(0)} + \sqrt{2}\cos\vartheta^{(0)}}{1 - \frac{1}{\sqrt{2}}\text{ctg}\vartheta^{(0)}}.\end{aligned}\quad (67)$$

The parameter ϵ_0 can be expressed in terms of physical meson masses by applying Dashen's theorem [14], which implies the identity of the pion and kaon electromagnetic mass shifts up to $\mathcal{O}(e^2p^2)$

$$\epsilon_0 = \frac{m_{K^0}^2 - m_{K^\pm}^2 + m_{\pi^\pm}^2 - m_{\pi^0}^2}{\sqrt{3}(m_\eta^2 - m_\pi^2)}.\quad (68)$$

There have been estimates in the literature that Dashen's theorem is significantly violated at higher orders due to chiral symmetry breaking effects [15, 16, 17]. On the other hand, a recent non-perturbative approach to the hadronic decays of η and η' indicated that higher order corrections to this low-energy theorem may be small [18]. In any case, the isospin-violating effects in the decay widths constitute a small correction, so that it is safe to employ Dashen's theorem in the present work.

This time the fit to the central experimental values yields

$$\begin{aligned}B_\pi/F_\pi &= -0.134 \text{ GeV}^{-1}, \\ B_\eta/F_\eta^8 &= -0.0598 \text{ GeV}^{-1}, \\ B_{\eta'}/F_{\eta'}^0 &= -0.208 \text{ GeV}^{-1}.\end{aligned}\quad (69)$$

and setting $F_{\eta'}^0 = 1.07F_\pi$ in the ratio $B_{\eta'}^{WZW}/F_{\eta'}^0$ reproduces the fitted value.

It should be emphasized that we fitted our results to the current world average value for $\Gamma_{\pi^0 \rightarrow \gamma\gamma} = 7.74 \pm 0.55 \text{ eV}$ [20]. On the other hand, it is possible to estimate in a model-dependent way the size of the counter term contributions. In [21], *e.g.*, the values of the counter term contributions to the π^0 decay have been estimated by means of a QCD sum rule for the general three-point function involving the pseudoscalar density and two vector currents. Within that approach the authors find a slightly enhanced width of $\Gamma_{\pi^0 \rightarrow \gamma\gamma} = 8.10 \pm 0.08 \text{ eV}$.

A comparison of the numerical results for $N_c = 3$ and $N_c = 5$ and setting all non-anomalous contact terms of unnatural parity to zero yields $\Gamma_{\pi^0 \rightarrow \gamma\gamma}^{N_c=3} = 8.00 \text{ eV}$, $\Gamma_{\pi^0 \rightarrow \gamma\gamma}^{N_c=5} = 8.18 \text{ eV}$ for

the pion decay, $\Gamma_{\eta' \rightarrow \gamma\gamma}^{N_c=3} = 0.456$ keV, $\Gamma_{\eta' \rightarrow \gamma\gamma}^{N_c=5} = 0.920$ keV for the η and $\Gamma_{\eta' \rightarrow \gamma\gamma}^{N_c=3} = 13.7$ keV, $\Gamma_{\eta' \rightarrow \gamma\gamma}^{N_c=5} = 4.28$ keV for the η' . Again, $N_c = 5$ is ruled out by the η and η' decays, but no rigorous statement can be made for the π^0 , although the value for $N_c = 3$ is in better agreement with the current world average.

5 Conclusions

In the present work, we investigated the two-photon decays of π^0 , η and η' in the combined $1/N_c$ and chiral expansions. The cancellation of triangle anomalies in the standard model requires the quark charges to depend on N_c . We have shown that the WZW term of the $U(3)$ effective theory decomposes into the conventional anomalous $SU(3)$ WZW Lagrangian, a Goldstone-Wilczek term and counter terms of unnatural parity which involve the singlet field η_0 . The independence of the π^0 and η decay amplitudes on N_c which was shown in [2] to occur at tree-level due to partial cancellations of the WZW term with a Goldstone-Wilczek term, persists at one-loop order, although the vertices of the pertinent loop graphs *do* exhibit an N_c dependence.

We performed a one-loop calculation including counter terms up to next-to-next-to leading order in large N_c ChPT in which η - η' mixing has also been taken into account up to one-loop order. Within the bookkeeping of large N_c ChPT, the leading contribution to the η decay arises from mixing with the η' .

From a fit to the experimental decay widths and under the assumption that higher orders beyond our working precision can be neglected it follows that contributions from the counter terms are small. Since the cancellation of Witten's global $SU(2)_L$ anomaly requires N_c to be odd and a world with $N_c = 1$ has no strong interactions, we compare the cases $N_c = 3$ and $N_c = 5$. The numerical results of the η and η' decay widths for $N_c = 3$ are close to the experimental values and clearly rule out the case $N_c = 5$. We have furthermore given an estimate on the N_c dependence of the pion decay due to different up- and down-quark masses by taking π^0 - η - η' mixing at leading order into account. The N_c dependence of the π^0 decay is smaller than the experimental uncertainty and is therefore not suited to extract the number of colors. We conclude that both the η and the η' decay show clear evidence that we live in a world with three colors.

It has been pointed out in [2] that at tree level the process $\eta \rightarrow \pi^+\pi^-\gamma$ is proportional to N_c^2 and should replace the textbook process $\pi^0 \rightarrow \gamma\gamma$ lending support to $N_c = 3$. It will be thus of interest to investigate the decays $\eta \rightarrow \pi^+\pi^-\gamma$ and $\eta' \rightarrow \pi^+\pi^-\gamma$ within the framework of large N_c ChPT [23].

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A Scaling behavior of the coupling constants

In this appendix, we derive the scaling behavior of the coupling constants which contribute to the decays. The transformation properties of the constants $\tilde{L}_{2,3}$ have already been given in [6]

and we merely quote the result here. The quantities \tilde{L}_2 and \tilde{L}_3 are renormalized according to

$$\tilde{L}_2^{ren} = Z_A \tilde{L}_2 - \kappa, \quad \tilde{L}_3^{ren} = Z_A \tilde{L}_3 - \kappa \quad (\text{A.1})$$

with

$$\kappa = \frac{N_c(Z_A - 1)}{144\pi^2}. \quad (\text{A.2})$$

Since the singlet field $\bar{\psi}$ scales as $\bar{\psi}^{ren} = Z_A^{-1}\bar{\psi}$, the scaling behavior of $\tilde{L}_{2,3}^{ren}$ ensures that the Lagrangian $\mathcal{L}_{WZW} + \tilde{\mathcal{L}}_{p^4}^{(2)}$ remains invariant to order δ^2 under changes of the QCD running scale. In order to study the transformation properties of $\tilde{L}_{5,6}$ we rewrite $\tilde{\mathcal{L}}_{p^4}^{(3)}$

$$\begin{aligned} \tilde{\mathcal{L}}_{p^4}^{(3)} &= 2\tilde{L}_5 \bar{\psi} \left(\langle F_r \rangle \langle F_r \rangle + \langle F_l \rangle \langle F_l \rangle \right) + 2\tilde{L}_6 \bar{\psi} \langle F_r \rangle \langle F_l \rangle \\ &= 2\bar{\psi} [2\tilde{L}_5 + \tilde{L}_6] \langle dv \rangle \langle dv \rangle + 2\bar{\psi} [2\tilde{L}_5 - \tilde{L}_6] \langle da \rangle \langle da \rangle \end{aligned} \quad (\text{A.3})$$

This yields the transformation properties

$$\begin{aligned} (2\tilde{L}_5 + \tilde{L}_6)^{ren} &= Z_A(2\tilde{L}_5 + \tilde{L}_6), \\ (2\tilde{L}_5 - \tilde{L}_6)^{ren} &= Z_A^3(2\tilde{L}_5 - \tilde{L}_6) - \frac{N_c(Z_A^3 - 1)}{432\pi^2}, \end{aligned} \quad (\text{A.4})$$

so that the Lagrangian $\mathcal{L}_{WZW} + \tilde{\mathcal{L}}_{p^4}^{(2)} + \tilde{\mathcal{L}}_{p^4}^{(3)}$ remains renormalization group invariant.

On the other hand, the contact terms \tilde{W}_1 and \tilde{W}_2 in the Lagrangian of sixth chiral order can be written as

$$i\tilde{W}_1(\bar{\psi})e^{\frac{i}{3}\bar{\psi}}\langle \bar{U}\chi^\dagger F_{\bar{r}}^2 + \chi^\dagger \bar{U} F_l^2 \rangle + i\tilde{W}_2(\bar{\psi})e^{\frac{i}{3}\bar{\psi}}\langle \chi^\dagger F_{\bar{r}} \bar{U} F_l + \bar{U} \chi^\dagger \bar{U} F_l \bar{U}^\dagger F_{\bar{r}} \rangle + h.c. + \dots \quad (\text{A.5})$$

The ellipsis in Eq. (A.5) denotes terms with more than one flavor trace which involve contributions from other contact terms and are irrelevant for the discussion of the scaling behavior of \tilde{W}_1 and \tilde{W}_2 . From (A.5) we obtain the transformation properties

$$\begin{aligned} \tilde{W}_1(x)^{ren} &= \tilde{W}_1(Z_A x) e^{\frac{i}{3}(Z_A - 1)x}, \\ \tilde{W}_2(x)^{ren} &= \tilde{W}_2(Z_A x) e^{\frac{i}{3}(Z_A - 1)x}. \end{aligned} \quad (\text{A.6})$$

Expanding the potentials \tilde{W}_i in the singlet field $\bar{\psi}$ yields for the two leading expansion coefficients

$$\begin{aligned} (\tilde{w}_i^{(0)})^{ren} &= \tilde{w}_i^{(0)}, \\ (\tilde{w}_i^{(1)})^{ren} &= Z_A \tilde{w}_i^{(1)} + \frac{1}{3}(Z_A - 1)\tilde{w}_i^{(0)}, \quad i = 1, 2. \end{aligned} \quad (\text{A.7})$$

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